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# CONSTRUCTION OF MIXED ORTHOGONAL ARRAYS WITH HIGH STRENGTH

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A considerable portion of the work on mixed orthogonal arrays applies specifically to arrays of strength 2. Although strength $t = 2$ is arguably the most important case for statistical applications, there is an urgent need for better methods for $t \geq 3$. However, the knowledge on the existence of arrays for $t \geq 3$ is rather limited. In this paper, new construction methods for symmetric and asymmetric orthogonal arrays (OAs) with high strength are proposed by using lower strength orthogonal partitions of spaces and OAs. A positive answer is provided to the open problem in Hedayat, Sloane and Stufken (*Orthogonal Arrays: Theory and Applications* (1999) Springer) on developing better methods and tools for the construction of mixed orthogonal arrays with strength $t \geq 3$. Not only are the methods straightforward, but also they are useful for constructing symmetric or asymmetric OAs of arbitrary strengths, numbers of levels and various sizes. The constructed OAs can be utilized to generate more OAs. The resulting OAs have a high degree of flexibility and many other desirable properties. Some selective OAs are tabulated for practical uses.

**1. Introduction.** An orthogonal array (OA) OA$(N, p_1^{n_1} p_2^{n_2} \cdots p_v^{n_v}, t)$ is an array of size $N \times n$, where $n = n_1 + n_2 + \cdots + n_v$ is the total number of factors; the first $n_1$ columns have symbols from $\{0, \dots, p_1 - 1\}$, the next $n_2$ columns have symbols from $\{0, \dots, p_2 - 1\}$ and so on, with the property that in any $N \times t$ subarray, every possible $t$-tuple occurs an equal number of times as a row. If $p_1 = \cdots = p_v$, the OA is said to be a fixed or symmetric OA; otherwise, it is a mixed or asymmetric OA. If $t \geq 3$, the OA is said to be of high strength. For convenience and simplicity, a symmetric OA of strength $t$ with $p$ levels from the ring $Z_p$ is denoted by OA$(N, p^n, t)$. An OA that achieves the Rao bound on the number of runs is said to be *tight* (Hedayat, Sloane and Stufken (1999)).

Chêng (1980) provided a precise statement and rigorous proof of the universal optimality of an OA with variable numbers of symbols as a fractional factorial design. OAs of strength 2 have been studied extensively. A great deal of methods and results can be found in the monograph (Hedayat, Sloane and Stufken (1999)), the handbook (Colbourn and Dinitz (2007)), and other literature (Hedayat, Stufken and Su (1996), Pang (2004), Zhang (2006, 2007), Zhang, Lu and Pang (1999) and Zhang, Pang and Wang (2001)). In comparison with those of strength 2, little is known about OAs of high strength ($t \geq 3$). Generally, finding OAs of high strength is more challenging than finding OAs of strength 2, but they are more useful than OAs of strength 2 in many areas (Carlet and Chen (2018), Colbourn and Dinitz (2007), Kuhfeld (2018) and Pang et al. (2018)), such as $k$-multipartite maximally entangled states (Goyeneche and Życzkowski (2014)). Construction of these states is an important open and well-known hard problem with ramifications in the theory of quantum information (Lo, Curty

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and Qi (2012), Riebe et al. (2004) and Zhao et al. (2004)). Pang et al. (2019) answered the open problem and obtained two and three-uniform states of almost every $N$ qudits from OAs. Although $\mathrm{OA}(N, v^k, t)$'s with $N \le 3^{132}$ or $t \le 32$ and asymmetric OAs with sizes $\le 4^{24}$ used in computer science, coding theory (Bierbrauer (2005) and Stinson (2004)), and cryptography have been partly obtained in Aggarwal and Budhraja (2002), the application of OAs with larger parameters is rather limited because of their scarcity. Hedayat, Sloane and Stufken (1999) proposed Research Problem 9.33: develop better methods and tools for the construction of mixed orthogonal arrays with strength $t \ge 3$. Besides orthogonality, OAs of high strength have projection properties of high order and uniformity (Dean et al. (2015), He and Tang (2014), Lin, Mukerjee and Tang (2009), Lin et al. (2010), Mukerjee, Sun and Tang (2014) and Liu and Liu (2015)), which are employed in theoretical studies and computer experiments (Sun and Tang (2017) and Tang (1993)). Therefore, some statisticians and mathematicians are devoted to constructing OAs of high strength (Hedayat, Stufken and Su (1996), Ji and Yin (2010), Schoen, Eendebak and Nguyen (2010), Suen, Das and Dey (2001), Suen and Dey (2003) and Yin et al. (2011)). On the other hand, many other designs related to OAs have been introduced, for example, strong OAs (He and Tang (2013, 2014)), covering arrays (Ji and Yin (2010) and Yin et al. (2011)), nearly OAs (Wang and Wu (1992)), mappable nearly OAs (Mukerjee, Sun and Tang (2014)), compound OAs (Hedayat, Sloane and Stufken (1999)), and augmented OAs (Stinson (2018)). There are many challenging unsolved mathematical and statistical problems in this area.

In the data science era, high strength OAs of large size are indispensable. Unfortunately, little consideration has been given to the construction of asymmetric OAs with strength greater than two. Therefore, there is a need for the construction of high strength OAs $\mathrm{OA}(N, p_1^{n_1} p_2^{n_2} \cdots p_v^{n_v}, t)$, especially with some nonprime power $p_i$'s. In this paper, some new construction methods of symmetric and asymmetric OAs of high strength are proposed by using an initial OA with strength $t \ge 1$, and orthogonal partitions of spaces and OAs. Not only are they straightforward, but also they can be used to construct symmetric or asymmetric OAs with various strengths, larger sizes and flexibility in the choice of factor levels. Moreover, since Theorems 3.1 and 4.1 (as will be shown) do not rely on difference schemes and finite fields, we can provide OAs having factors whose numbers of levels are nonprime powers, such as $\mathrm{OA}(2^{2n+5}3^2, 2^3 4^n 12^2, n+3)$ for $n \ge 2$, $\mathrm{OA}(4^{s_1+1}9^{s_2}p^2, 2^4 p^2 (2^{s_1})^2 (3^{s_2})^2, 5)$ with an even $p$. Additionally, using orthogonal partitions of symmetric and asymmetric OAs enables us to obtain some new infinite families of high strength OAs under certain conditions, such as $\mathrm{OA}(Np^{t-1}, p^m p_2^{n_2} \cdots p_v^{n_v}, t)$ and $\mathrm{OA}(2^5 p_1 p_2 p_3, 2^{18} p_1^1 p_2^1 p_3^1, 3)$. Some existing classes of tight arrays and arrays with the maximal numbers of columns can be obtained as special cases. As a consequence, we provide a positive answer to the open Research Problem 9.33 in Hedayat, Sloane and Stufken (1999) on developing better methods and tools for the construction of mixed orthogonal arrays.

The remainder of this paper is organized as follows. Section 2 introduces some notation, and basic concepts of orthogonal partition, as well as some lemmas useful in this work. Section 3 proposes methods primarily for constructing asymmetric OAs using $(r+1)$-column initial OAs of strength $r$. In Section 4, we study some extended constructions of asymmetric OAs with relatively more columns for the same strength compared with the arrays in Section 3. Sections 3 and 4 also provide an illustrative example and a construction procedure for each theorem. Section 5 draws some conclusions. Some newly obtained OAs and families of OAs of high strength are listed in Tables 1 and 2. And matrix forms of a subset of the arrays are displayed on the website http://web.stat.nankai.edu.cn/mqliu/MOA/MixedOA.html. A numerical verification confirms that these arrays are indeed correct. All proofs of the lemmas and theorems can be found in Part II in the Supplementary Material (Pang et al. (2021)).

**2. Preliminaries.** To present our results, we first make some preparations. Let $Z_p^n$ denote the $n$-dimensional space over a ring $Z_p = \{0, 1, \dots, p-1\}$. If necessary, a space $Z_{p_1}^{n_1} \times Z_{p_2}^{n_2} \times \cdots \times Z_{p_k}^{n_k}$ can also be seen as an $\mathrm{OA}(\prod_{i=1}^k p_i^{n_i}, p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k}, \sum_{i=1}^k n_i)$. Let $A^T$ denote the transpose of the matrix $A$ and $R_p = (0, 1, \dots, p-1)^T$. $0_r$ and $1_r$ represent $r \times 1$ vectors of 0's and 1's, respectively. $I_r$ is the identity matrix of order $r$. The Kronecker product $\otimes$ and Kronecker sum $\oplus$ are defined, respectively, as $B \otimes C = (b_{ij}C)_{su \times tv}$ and $B \oplus C = (b_{ij} + C)_{su \times tv}$ if $B = (b_{ij})_{s \times t}$ and $C_{u \times v}$ are based on $Z_p$. Let $K(n, m) = \sum_{i=1}^n \sum_{j=1}^m [(e_i(n) e_j^T(m)) \otimes (e_j(m) e_i^T(n))]$ be a permutation matrix as in Zhang, Pang and Wang (2001), where $e_i(n) = (\underbrace{0, \dots, 0}_{i-1}, 1, \underbrace{0, \dots, 0}_{n-i})^T$.

DEFINITION 2.1. Let $A$ be an $\mathrm{OA}(N, p_1 \cdots p_n, t)$. Suppose the rows of $A$ can be partitioned into $s$ submatrices $A_0, \dots, A_{s-1}$ such that each $A_i$ is an $\mathrm{OA}(N/s, p_1 \cdots p_n, t_1)$ with $t_1 \geq 0$. Then the set $\{A_0, A_1, \dots, A_{s-1}\}$ is called an orthogonal partition of strength $t_1$ of $A$. In particular, $\{A_0, A_1, \dots, A_{s-1}\}$ is said to be a strength $t_1$ orthogonal partition of a space $Z_p^n$ if $A = Z_p^n$.

By using permutation properties of the matrix $K(n, m)$, we present several indispensable lemmas.

LEMMA 2.1. *Assume that there are two sets $\{A_{11}, \dots, A_{k1}\}$ of $m \times m_1$ matrices and $\{A_{12}, \dots, A_{k2}\}$ of $n \times n_1$ matrices, with the property that an $m_1$-tuple $x$ occurs $u$ times as a row in each of the matrices $A_{i1}$, and an $n_1$-tuple $y$ occurs $v$ times as a row in all the matrices $A_{12}, A_{22}, \dots, A_{k2}$. Then there are $uv$ rows $(x, y)$ and $uv$ rows $(y, x)$ in $M_1$ and $M_2$, respectively, where*

$$M_1 = \begin{pmatrix} A_{11} \otimes 1_n & 1_m \otimes A_{12} \\ A_{21} \otimes 1_n & 1_m \otimes A_{22} \\ \cdots & \cdots \\ A_{k1} \otimes 1_n & 1_m \otimes A_{k2} \end{pmatrix} \quad \text{and} \quad M_2 = \begin{pmatrix} A_{12} \otimes 1_m & 1_n \otimes A_{11} \\ A_{22} \otimes 1_m & 1_n \otimes A_{21} \\ \cdots & \cdots \\ A_{k2} \otimes 1_m & 1_n \otimes A_{k1} \end{pmatrix}.$$

LEMMA 2.2. *Let $A$ be a matrix of $m$ rows. Then*

$$K(n, m)(A \otimes 1_n) = 1_n \otimes A, \quad \text{and} \quad K(m, n)(1_n \otimes A) = A \otimes 1_n.$$

LEMMA 2.3. *Let $a_j$ be an $m_j \times 1$ vector for $j = 1, 2, 3$. Then*

$$\begin{aligned} &(K(m_2, m_1) \otimes I_{m_3})(a_1 \otimes 1_{m_2 m_3}, 1_{m_1} \otimes a_2 \otimes 1_{m_3}, 1_{m_1 m_2} \otimes a_3) \\ &\quad = (1_{m_2} \otimes a_1 \otimes 1_{m_3}, a_2 \otimes 1_{m_1 m_3}, 1_{m_1 m_2} \otimes a_3). \end{aligned}$$

REMARK 2.1. When the vector $a_j$ is replaced by a matrix $A_j$, Lemma 2.3 also holds.

LEMMA 2.4. *Let $A_{ij}$ be an $n_j \times 1$ vector for $1 \leq i \leq k$ and $1 \leq j \leq 3$. Suppose that a pair $(x_1, x_3)$ occurs $v$ times as a row in a matrix $B$ below and an element $x_2$ occurs $u$ times in each $A_{i2}$. Then the triple $(x_1, x_2, x_3)$ appears $uv$ times as a row in $H$. Here,*

$$B = \begin{pmatrix} A_{11} \otimes 1_{n_3} & 1_{n_1} \otimes A_{13} \\ A_{21} \otimes 1_{n_3} & 1_{n_1} \otimes A_{23} \\ \cdots & \cdots \\ A_{k1} \otimes 1_{n_3} & 1_{n_1} \otimes A_{k3} \end{pmatrix},$$

*and*

$$H = \begin{pmatrix} A_{11} \otimes 1_{n_2 n_3} & 1_{n_1} \otimes A_{12} \otimes 1_{n_3} & 1_{n_1 n_2} \otimes A_{13} \\ A_{21} \otimes 1_{n_2 n_3} & 1_{n_1} \otimes A_{22} \otimes 1_{n_3} & 1_{n_1 n_2} \otimes A_{23} \\ \cdots & \cdots & \cdots \\ A_{k1} \otimes 1_{n_2 n_3} & 1_{n_1} \otimes A_{k2} \otimes 1_{n_3} & 1_{n_1 n_2} \otimes A_{33} \end{pmatrix}.$$

REMARK 2.2. Lemma 2.4 also holds when $A_{ij}$ is an $n_j \times m_j$ matrix and $x_j$ is replaced by a $1 \times m_j$ vector, where $1 \le i \le k$ and $1 \le j \le 3$.

Note that these lemmas will be used to compute the number of times a $t$-tuple appears as a row in an OA of strength $t$; the lemmas are the backbone of our principal results. Sections 3 and 4 will propose new methods primarily for constructing asymmetric OAs from an initial OA of strength $r$ based on these lemmas and orthogonal partitions. Section 3 concerns cases using the initial OA with $r+1$ columns, while Section 4 concerns cases using the initial OA with $q$ $(> r+1)$ columns.

## 3. Construction of asymmetric OAs using an $(r+1)$-column initial OA of strength $r$.

The objective of this section is to present the constructions of asymmetric OAs with high strength by applying the above lemmas and orthogonal partitions of spaces and OAs. In particular, some of the run sizes of the resulting OAs are equal to the product of the top $t$ highest numbers of levels since the minimum run size of an OA with strength $t$ is required to be the least common multiple of products of any $t$ different numbers of levels. Some examples of new infinite families of OAs are presented. Since OAs with two or three levels are of particular interest to statistical applications, our arrays include primarily two or three levels along with several higher levels.

For simplicity in the following descriptions, we introduce some notation. For a fixed positive integer $j$, let $\{A_{l_1 j j}, A_{l_{2j} j}, \ldots, A_{l_{kj} j}\}$ be a set of $u \times v$ matrices. Three matrices

$$\begin{pmatrix} A_{l_{1j} j} \otimes 1_n \\ A_{l_{2j} j} \otimes 1_n \\ \cdots \\ A_{l_{kj} j} \otimes 1_n \end{pmatrix}, \qquad \begin{pmatrix} 1_m \otimes A_{l_{1j} j} \\ 1_m \otimes A_{l_{2j} j} \\ \cdots \\ 1_m \otimes A_{l_{kj} j} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1_m \otimes A_{l_{1j} j} \otimes 1_n \\ 1_m \otimes A_{l_{2j} j} \otimes 1_n \\ \cdots \\ 1_m \otimes A_{l_{kj} j} \otimes 1_n \end{pmatrix}$$

can be denoted by the symbols

$$(A_{[l_1,l_2,\ldots,l_k]j}, n), (m, A_{[l_1,l_2,\ldots,l_k]j}) \quad \text{and} \quad (m, A_{[l_1,l_2,\ldots,l_k]j}, n),$$

respectively, for positive integers $m$ and $n$.

THEOREM 3.1. *Let $L = (l_{ij})$ be an initial* $\mathrm{OA}(h, s_1 s_2 \cdots s_{r+1}, r)$ *with strength $r \ge 1$. Suppose that there exists an $n_j$-dimensional space $Z_{p_j}^{n_j}$ with an orthogonal partition of strength $t_j$, namely $\{A_{0j}, A_{1j}, \ldots, A_{(s_j-1)j}\}$, where $s_j | p_j^{n_j}$ and $t_j \ge 0$, for $1 \le j \le r+1$. Then the array $M_{r+1} = (A_1, A_2, \ldots, A_{r+1})$ is an* $\mathrm{OA}(h \prod_{j=1}^{r+1} (p_j^{n_j}/s_j), p_1^{n_1} \cdots p_{r+1}^{n_{r+1}}, t)$, *where $t = r + \sum_{j=1}^{r+1} t_j$ and $A_j = (\prod_{k=1}^{j-1} (p_k^{n_k}/s_k), A_{[l_1,l_2,\ldots,l_h]j}, \prod_{k=j+1}^{r+1} (p_k^{n_k}/s_k))$.*

As aforementioned, accompanying each theorem is an algorithm, and an example with more details to illustrate the construction procedure. According to Theorem 3.1, Algorithm 3.1 below is provided for constructing an $\mathrm{OA}(N, p_1^{n_1} \cdots p_v^{n_v}, t)$.

ALGORITHM 3.1. *Step 1.* According to the $v$ numbers of levels $p_1, \ldots, p_v$ in the desired OA, set the number of factors $r+1 = v$ and strength $r = v-1$ for an initial OA $L$.

*Step 2.* For $j = 1, \dots, r+1$, specify space $Z_{p_j}^{n_j}$ to be partitioned based on the parameters $p_j$ and $n_j$, and find an orthogonal partition $\{A_{0j}, \dots, A_{(s_j-1)j}\}$ of strength $t_j$ of $Z_{p_j}^{n_j}$ such that $r + \sum_{j=1}^{r+1} t_j = t$. Let the number of levels of the $j$th factor of $L$ be $s_j$, that is, $L = \mathrm{OA}(h, s_1 \cdots s_{r+1}, r)$.

*Step 3.* Place all of the orthogonal partitions $\{A_{0j}, \dots, A_{(s_j-1)j}\}$ into the array $M_{r+1} = (A_1, A_2, \dots, A_{r+1})$ in Theorem 3.1.

To increase the numbers of factors for the constructed OAs, we can search for OAs to partition orthogonally. The following theorem extends $Z_p^n$ in Theorem 3.1 to an $\mathrm{OA}(N, m_1 \cdots m_v, t)$, resulting in the arrays thus obtained having higher saturation percentages.

THEOREM 3.2. *Let $L = (l_{ij})$ be an initial* $\mathrm{OA}(h, s_1 \cdots s_{r+1}, r)$ *with strength $r \geq 1$. Let $u \in \{0, 1, \dots, r\}$ be a given integer. Suppose that there exists an $n_\alpha$-dimensional space $Z_{p_\alpha}^{n_\alpha}$ having an orthogonal partition with $s_\alpha$ blocks of strength $t_\alpha \geq 0$ for each $\alpha \in \{1, \dots, u\}$. Further, suppose there exists an orthogonal partition with $s_\beta$ blocks of strength $t_\beta \geq 0$ of* $\mathrm{OA}(N_\beta, m_{1\beta} \cdots m_{v_\beta\beta}, t)$ *for any $\beta \in \{u+1, \dots, r+1\}$ such that $t = r + \sum_{j=1}^{r+1} t_j$. Then there exists an* $\mathrm{OA}(h \prod_{\alpha=1}^{u} (p_\alpha^{n_\alpha}/s_\alpha) \prod_{\beta=u+1}^{r+1} (N_\beta/s_\beta), p_1^{n_1} \cdots p_u^{n_u} m_{1(u+1)} \cdots m_{v_{(r+1)}(r+1)}, t)$.

Our procedure for constructing an $\mathrm{OA}(N, p_1^{n_1} \cdots p_v^{n_v}, t)$ using Theorem 3.2 involves the following three steps.

ALGORITHM 3.2. *Step 1.* For a fixed $r \leq t-1$ and each $i = 1, \dots, v$, decompose $n_i = \sum_{j=1}^{r+1} n_{ij}$ such that there exists an $\mathrm{OA}(N_j, p_1^{n_{1j}} \cdots p_v^{n_{vj}}, t)$, and its an orthogonal partition of strength $t_j$, or there exists an orthogonal partition of strength $t_j$ of a space $Z_{p_1}^{n_{1j}} \times \cdots \times Z_{p_v}^{n_{vj}}$, uniformly denoted by $\{A_{0j}, \dots, A_{(s_j-1)j}\}$, satisfying $t = r + \sum_{j=1}^{r+1} t_j$ and making $s_j$ as large as possible for $j = 1, \dots, r+1$.

*Step 2.* Take an initial $\mathrm{OA}(h, s_1 \cdots s_{r+1}, r)$ according to all $s_j$'s.

*Step 3.* Substitute all of the orthogonal partitions $\{A_{0j}, \dots, A_{(s_j-1)j}\}$ into $A_1, A_2, \dots, A_{r+1}$ in the proof of Theorem 3.2 to produce the desired OA $M_{r+1}$.

The above theorems can not only construct new infinite families of OAs with various strengths, larger sizes and flexibility in the choice of factor levels, but also give an insight into the structure of the obtained OAs. Especially, Theorem 3.2 can be used for another new method for the construction of OAs (iterative method, see Section 5). In the following examples, new OAs can be obtained by taking the orthogonal partitions of the spaces and OAs and carefully choosing the initial $\mathrm{OA}(h, s_1 \cdots s_q, t)$ such that $h/(s_1 \cdots s_q)$ is as small as possible. In Example 3.1, the run size of the constructed OAs is equal to the product of the top $n+3$ highest numbers of levels, that is, $2^{2n+5}3^2 = 2^1 4^n 12^2$. The constructed OAs in Example 3.2 have more flexibility in the choice of parameters, such as the size, number of factors, number of levels and strength. Here, we utilize the following example to illustrate the application of Theorem 3.1.

EXAMPLE 3.1 (Construction of a new family of $\mathrm{OA}(2^{2n+5}3^2, 2^3 4^n 12^2, n+3)$ for $n \geq 2$). Step 1. According to the three different numbers of levels, that is, 2, 4 and 12 in the desired OA, set an initial OA $L$ with three factors and strength 2.

Step 2. Specify and partition spaces $Z_2^3$, $Z_4^n$ and $Z_{12}^2$.

We can decompose $Z_2^3$ into an orthogonal partition of strength 1. First, find a strength 1 $\mathrm{OA}(2, 2^3, 1)$, denoted by $A_{01} = (a_1, a_2, a_3)$. For instance, let $a_1 = a_2 = a_3 = R_2$. Second, take $A_{11} = (a_1, a_2, 1 + a_3)$, $A_{21} = (a_1, 1 + a_2, a_3)$ and $A_{31} = (a_1, 1 + a_2, 1 + a_3)$, where $+$ is the addition on $Z_2$. Then $\{A_{01}, \dots, A_{31}\}$ is the orthogonal partition we need.

Similarly, let $A_{02} = (b_1, \dots, b_n) = \mathrm{OA}(4^{n-1}, 4^n, n-1)$. Take $A_{i2} = (b_1, \dots, b_{n-1}, i + b_n)$, where $+$ is the addition on $Z_4$ and $i \in Z_4$. Then $\{A_{02}, \dots, A_{32}\}$ is an orthogonal partition of strength $n - 1$ of $Z_4^n$.

In the same manner, we can find an orthogonal partition $\{B_i | B_i = (R_{12}, i + R_{12}), i \in Z_{12}\}$ of strength 1 of $Z_{12}^2$. Let $A_{i3} = (B_{3i}^T, B_{3i+1}^T, B_{3i+2}^T)^T$ for $i \in Z_4$. Then $\{A_{03}, \dots, A_{33}\}$ is the orthogonal partition of strength 1 of $Z_{12}^2$.

Now, $t = 2 + 1 + n - 1 + 1 = n + 3$. Let the number of levels of each factor of the initial OA be 4. Choose an initial $\mathrm{OA}(16, 4^3, 2)$ as

$$L = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 \\ 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 & 1 & 0 & 3 & 2 & 2 & 3 & 0 & 1 & 3 & 2 & 1 & 0 \end{pmatrix}^T.$$

Step 3. From the three sets $\{A_{0j}, \dots, A_{3j}\}$, $j = 1, 2, 3$, we can obtain $A_1$, $A_2$ and $A_3$. Then the desired OA

$$M_3 = (A_1, A_2, A_3) = \begin{pmatrix}
A_{01} \otimes 1_{4^n \cdot 9} & 1_2 \otimes A_{02} \otimes 1_{36} & 1_{2 \cdot 4^{n-1}} \otimes A_{03} \\
A_{01} \otimes 1_{4^n \cdot 9} & 1_2 \otimes A_{12} \otimes 1_{36} & 1_{2 \cdot 4^{n-1}} \otimes A_{13} \\
A_{01} \otimes 1_{4^n \cdot 9} & 1_2 \otimes A_{22} \otimes 1_{36} & 1_{2 \cdot 4^{n-1}} \otimes A_{23} \\
A_{01} \otimes 1_{4^n \cdot 9} & 1_2 \otimes A_{32} \otimes 1_{36} & 1_{2 \cdot 4^{n-1}} \otimes A_{33} \\
A_{11} \otimes 1_{4^n \cdot 9} & 1_2 \otimes A_{02} \otimes 1_{36} & 1_{2 \cdot 4^{n-1}} \otimes A_{13} \\
A_{11} \otimes 1_{4^n \cdot 9} & 1_2 \otimes A_{12} \otimes 1_{36} & 1_{2 \cdot 4^{n-1}} \otimes A_{03} \\
A_{11} \otimes 1_{4^n \cdot 9} & 1_2 \otimes A_{22} \otimes 1_{36} & 1_{2 \cdot 4^{n-1}} \otimes A_{33} \\
A_{11} \otimes 1_{4^n \cdot 9} & 1_2 \otimes A_{32} \otimes 1_{36} & 1_{2 \cdot 4^{n-1}} \otimes A_{23} \\
A_{21} \otimes 1_{4^n \cdot 9} & 1_2 \otimes A_{02} \otimes 1_{36} & 1_{2 \cdot 4^{n-1}} \otimes A_{23} \\
A_{21} \otimes 1_{4^n \cdot 9} & 1_2 \otimes A_{12} \otimes 1_{36} & 1_{2 \cdot 4^{n-1}} \otimes A_{33} \\
A_{21} \otimes 1_{4^n \cdot 9} & 1_2 \otimes A_{22} \otimes 1_{36} & 1_{2 \cdot 4^{n-1}} \otimes A_{03} \\
A_{21} \otimes 1_{4^n \cdot 9} & 1_2 \otimes A_{32} \otimes 1_{36} & 1_{2 \cdot 4^{n-1}} \otimes A_{13} \\
A_{31} \otimes 1_{4^n \cdot 9} & 1_2 \otimes A_{02} \otimes 1_{36} & 1_{2 \cdot 4^{n-1}} \otimes A_{33} \\
A_{31} \otimes 1_{4^n \cdot 9} & 1_2 \otimes A_{12} \otimes 1_{36} & 1_{2 \cdot 4^{n-1}} \otimes A_{23} \\
A_{31} \otimes 1_{4^n \cdot 9} & 1_2 \otimes A_{22} \otimes 1_{36} & 1_{2 \cdot 4^{n-1}} \otimes A_{13} \\
A_{31} \otimes 1_{4^n \cdot 9} & 1_2 \otimes A_{32} \otimes 1_{36} & 1_{2 \cdot 4^{n-1}} \otimes A_{03}
\end{pmatrix}.$$

For $n = 2$, an apparently new $\mathrm{OA}(2^9 3^2, 2^3 4^2 12^2, 5)$ can be constructed.

The following is an example of the use of Theorem 3.2.

EXAMPLE 3.2. Assume that an $\mathrm{OA}(N, p_2^{n_2} \cdots p_v^{n_v}, t)$ exists for $t \geq 3$. Let $p \geq t$ be a prime power and $p | N$. Then a new family of $\mathrm{OA}(N p^{t-1}, p^m p_2^{n_2} \cdots p_v^{n_v}, t)$ exists whenever $m = p + 1$ if $p$ is a power of 2 and $t = 3$, and $m = p$ otherwise.

Step 1. For a fixed $r = 1$, decompose $n_1 = m + 0$, $n_i = 0 + n_i$ for $i = 2, \dots, v$. From Theorems 3.1 and 3.2 and Property 7 on page 5 in Hedayat, Sloane and Stufken (1999), an $\mathrm{OA}(p^t, p^m, t)$ exists, and it can be divided into an orthogonal partition of strength $t - 1$, say, $\{A_{01}, \dots, A_{(p-1)1}\}$. Since $p | N$, there exists an orthogonal partition of strength 0 of the $\mathrm{OA}(N, p_2^{n_2} \cdots p_v^{n_v}, t)$, say $\{A_{02}, \dots, A_{(p-1)2}\}$.

Step 2. Let the initial OA be an $\mathrm{OA}(p, p^2, 1)$.

Step 3. From the orthogonal partitions $\{A_{0j}, \dots, A_{(p-1)j}\}$ for $j = 1, 2$, $A_1$ and $A_2$ can be generated easily. Then the desired OA $M_{r+1}$ follows.

Especially, when $t = 3$, $N = 48$, $p_2 = 3$, $n_2 = 1$, $p_3 = 2$, $n_3 = 9$ and $p = 16$, we can obtain an $\mathrm{OA}(2^{12} 3^1, 2^9 3^1 16^{17}, 3)$.

Let $t = 3$, $N = 40$, $p_2 = 5$, $n_2 = 1$, $p_3 = 2$, $n_3 = 6$ and $p = 8$, the example yields an $\mathrm{OA}(2^9 5^1, 2^6 5^1 8^9, 3)$.

For $t = 4$, $N = 144$, $p_2 = 3$, $n_2 = 1$, $p_3 = 2$ and $n_3 = 6$, with $p = 9$ and $p = 16$, we can construct an $\mathrm{OA}(2^4 3^8, 2^6 3^1 9^9, 4)$ and an $\mathrm{OA}(2^{16} 3^2, 2^6 3^1 16^{16}, 4)$, respectively.

In Examples 3.1 and 3.2, we can construct $M_1 = \mathrm{OA}(2^9 3^2, 2^3 4^2 12^2, 5)$ and $M_2 = \mathrm{OA}(2^9 5^1, 2^6 5^1 8^9, 3)$. As can be seen from the following comparison, these arrays are new and cannot be obtained using previous methods.

(a) Hedayat, Sloane and Stufken (1999) and Hedayat, Stufken and Su (1996) constructed symmetric OAs of high strength. Ji and Yin (2010) and Yin et al. (2011) proved the existence of symmetric OAs of strength 3. However, both $M_1$ and $M_2$ are asymmetric OAs.

(b) Suen, Das and Dey (2001) and Suen and Dey (2003) proposed a general method for mainly constructing asymmetric OAs of strengths 3 and 4. This method was later extended by Zhang, Deng and Dey (2017) and Zhang, Zong and Dey (2016), but they obtained only families of OAs with prime-power run sizes. However, neither of the run sizes of $M_1$ and $M_2$ is a prime power.

(c) Using difference schemes, Chen and Lei (2017) studied the construction of OAs with strength 3. $M_1$ has strength 5. $M_2$ cannot be obtained through the use of such a method, otherwise it can be written as the product of two arrays $\mathrm{OA}(2^9, 2^6 8^9, 3)$ and $\mathrm{OA}(5, 5^1, 3)$. However, the $\mathrm{OA}(2^9, 2^6 8^9, 3)$ does not exist as far as is currently known.

(d) It can be seen that the OAs constructed by Schoen, Eendebak and Nguyen (2010) have limited run sizes $\le 64$ for strength 3 and $\le 168$ for strength 4. However, it is obvious that the run sizes of both $M_1$ and $M_2$ are greater than 168.

(e) Jiang and Yin (2013) obtained a family of $\mathrm{OA}(n^t, p_1 \cdots p_k, t)$. Neither of the run sizes of $M_1$ and $M_2$ is a power of an integer.

(f) Neither $M_1$ nor $M_2$ can be obtained by the product construction method of Chen, Ji and Lei (2014).

The proposed methods are different from (a), (b) and (c), since they do not rely on the difference schemes and finite fields.

**4. Construction of asymmetric OAs using $q$ ($> r+1$)-column initial OA of strength $r$.** In this section, we study an extended construction of asymmetric OAs having more columns for the same strength than the arrays in Section 3. We will use initial arrays with strength $r$ that have more than $r + 1$ columns. The orthogonal partitions of spaces required for the proposed methods could be obtained using row permutations, and the orthogonal partitions of OAs can be obtained mainly using Property 7 on page 5 in Hedayat, Sloane and Stufken (1999) and the difference schemes in Hedayat, Stufken and Su (1996). Moreover, our theorems also imply that the new OAs obtained in this study have useful orthogonal partitions.

THEOREM 4.1. *Suppose that $L_{hq} = (l_{ij})$ is an initial $\mathrm{OA}(h, s_1 s_2 \cdots s_q, r)$ with $r \ge 1$ and $q > r + 1$. Further, suppose there exists an $n_j$-dimensional space $Z_{p_j}^{n_j}$ with an orthogonal partition of strength $t_j$, namely, $\{A_{0j}, A_{1j}, \dots, A_{(s_j-1)j}\}$, where $t_j \ge 0$ for $j = 1, 2, \dots, q$. Then we can construct an $\mathrm{OA}(h \prod_{j=1}^{q} (p_j^{n_j}/s_j), p_1^{n_1} p_2^{n_2} \cdots p_q^{n_q}, t)$ where $t = \min_{1 \le j_1 < j_2 < \cdots < j_{r+1} \le q} \{t_{j_1} + t_{j_2} + \cdots + t_{j_{r+1}} + r\}$.*

The following algorithm is performed to construct an $\mathrm{OA}(N, p_1^{n_1} \cdots p_v^{n_v}, t)$ in accordance with Theorem 4.1.

ALGORITHM 4.1. *Step 1.* Identify an initial OA with the number of factors $q = v$ and strength $r < v - 1$ according to the $v$ numbers of levels $p_1, \dots, p_v$ in the desired OA.

*Step 2.* For $j = 1, \dots, q$, specify space $Z_{p_j}^{n_j}$ to be partitioned in terms of the parameters $p_j$ and $n_j$, find an orthogonal partition $\{A_{0j}, \dots, A_{(s_j-1)j}\}$ of strength $t_j$ of $Z_{p_j}^{n_j}$, and select an $r$ such that $\min_{1\le j_1<\cdots<j_{r+1}\le q}\{t_{j_1} + \cdots + t_{j_{r+1}} + r\} = t$. Let the number of levels of the $j$th factor of the initial OA be $s_j$, $j = 1, \dots, q$. Take $L_{hq} = \mathrm{OA}(h, s_1 s_2 \cdots s_q, r)$.

*Step 3.* Place all of the orthogonal partitions $\{A_{0j}, \dots, A_{(s_j-1)j}\}$ into $A_1, \dots, A_q$ in the proof of Theorem 4.1 to produce the desired OA $M_q$.

Corollary 4.1 below immediately follows from Theorem 4.1.

COROLLARY 4.1. *Under the condition of Theorem* 4.1, *we can obtain an symmetric* $\mathrm{OA}(h\prod_{j=1}^{q}(p^{n_j}/s_j), p^{\sum_{j=1}^{q} n_j}, t)$, *if* $p_1 = \cdots = p_q = p$.

EXAMPLE 4.1 (A new family of $\mathrm{OA}(4^{s_1+1}9^{s_2}p^2, 2^4p^2(2^{s_1})^2(3^{s_2})^2, 5)$ constructed for an even $p$). The desired OA can be written as

$$\mathrm{OA}(N, p_1^{n_1} \cdots p_v^{n_v}, t) = \mathrm{OA}(4^{s_1+1}9^{s_2}p^2, 2^2 2^2 p^2 (2^{s_1})^2 (3^{s_2})^2, 5).$$

Step 1. According to the five numbers of levels 2, 2, $p$, $2^{s_1}$ and $3^{s_2}$ in the OA above, choose an initial OA with five factors and strength $r \le 3$.

Step 2. Specify spaces $Z_2^2$, $Z_2^2$, $Z_p^2$, $Z_{2^{s_1}}^2$ and $Z_{3^{s_2}}^2$.

Now, $\{A_{i1} | A_{i1} = (R_2, i + R_2), i \in Z_2\}$ is an orthogonal partition of strength 1 of $Z_2^2$. Let $A_{i2} = A_{i1}$ for $i \in Z_2$. Similarly, $\{A_{i3} | A_{i3} = (0_{p/2} \oplus R_p, ((p/2)i + R_{p/2}) \oplus R_p), i \in Z_2\}$ is an orthogonal partition of strength 1 of $Z_p^2$.

We can find the orthogonal partitions $\{A_{i4} | A_{i4} = (0_{2^{s_1-1}} \oplus R_{2^{s_1}}, (2^{s_1-1}i + R_{2^{s_1-1}}) \oplus R_{2^{s_1}}), i \in Z_2\}$ and $\{A_{i5} | A_{i5} = (0_{3^{s_2-1}} \oplus R_{3^{s_2}}, (3^{s_2-1}i + R_{3^{s_2-1}}) \oplus R_{3^{s_2}}), i \in Z_3\}$ of strength 1 of $Z_{2^{s_1}}^2$ and $Z_{3^{s_2}}^2$, respectively.

Since $r = 2$, we have $t = \min_{1\le j_1<j_2<j_3\le 5}\{2 + t_{j_1} + t_{j_2} + t_{j_3}\} = 2 + 1 + 1 + 1 = 5$ and then take $L_{hq} = \mathrm{OA}(12, 2^4 3^1, 2)$.

Step 3. Substitute the five orthogonal partitions $\{A_{0j}, A_{1j}\}$, $j = 1, 2, 3, 4$ and $\{A_{05}, A_{15}, A_{25}\}$ into the array $M_5 = (A_1, A_2, A_3, A_4, A_5)$.

Particularly, for $p = 2, 4, 6$, we can construct some apparently new $\mathrm{OA}(2^6 3^2, 2^8 3^2, 5)$, $\mathrm{OA}(2^8 3^2, 2^6 3^2 4^2, 5)$, $\mathrm{OA}(2^6 3^4, 2^6 3^2 6^2, 5)$, respectively.

By arguments similar to those of Theorem 3.2, the following theorem will extend $Z_p^n$ in Theorem 4.1 to an $\mathrm{OA}(N, m_1 \cdots m_v, t)$ to improve the saturation percentage of the constructed OA.

THEOREM 4.2. *Let $L_{hq} = (l_{ij})$ be an initial* $\mathrm{OA}(h, s_1 \cdots s_q, r)$ *with strength $r \ge 1$ and $q > r + 1$. Let $u \in \{0, 1, \dots, q - 1\}$ be a given integer. Suppose an $n_\alpha$-dimensional space $Z_{p_\alpha}^{n_\alpha}$ has an orthogonal partition with $s_\alpha$ blocks of strength $t_\alpha \ge 0$ for every $\alpha \in \{1, \dots, u\}$. Further, suppose there exists an orthogonal partition with $s_\beta$ blocks of strength $t_\beta \ge 0$ of* $\mathrm{OA}(N_\beta, m_{1\beta} \cdots m_{v_\beta \beta}, t)$ *for each $\beta \in \{u + 1, \dots, q\}$ such that $t = \min_{1\le j_1<\cdots<j_{r+1}\le q}\{t_{j_1} + \cdots + t_{j_{r+1}} + r\}$. Then an* $\mathrm{OA}(h\prod_{\alpha=1}^{u}(p_\alpha^{n_\alpha}/s_\alpha)\prod_{\beta=u+1}^{q}(N_\beta/s_\beta), p_1^{n_1} \cdots p_u^{n_u} m_{1(u+1)} \cdots \times m_{v_{(u+1)}(u+1)} \cdots m_{1q} \cdots m_{v_q q}, t)$ *exists.*

Based on Theorem 4.2, we introduce Algorithm 4.2 for the construction of an $\mathrm{OA}(N, p_1^{n_1} \cdots p_v^{n_v}, t)$ as follows.

ALGORITHM 4.2. *Step 1.* For a fixed $r$, $q$ $(> r + 1)$ and each $i = 1, \dots, v$, decompose $n_i = \sum_{j=1}^{q} n_{ij}$ such that there exists an $\mathrm{OA}(N_j, p_1^{n_{1j}} \cdots p_v^{n_{vj}}, t)$ and its an orthogonal partition of strength $t_j$, or there exists an orthogonal partition of strength $t_j$ of a space $Z_{p_1}^{n_{1j}} \times \cdots \times Z_{p_v}^{n_{vj}}$, uniformly denoted by $\{A_{0j}, \dots, A_{(s_j-1)j}\}$ that satisfies $t = \min_{1\le j_1<j_2<\cdots<j_{r+1}\le q}\{t_{j_1} + t_{j_2} + \cdots + t_{j_{r+1}} + r\}$ with $s_j$ as large as possible for $j = 1, \dots, q$.

*Step 2.* Take the initial $\mathrm{OA}(h, s_1 \cdots s_q, r)$ according to all $s_j$'s.

*Step 3.* Using all of the orthogonal partitions $\{A_{0j}, \dots, A_{(s_j-1)j}\}$, compute $A_1, \dots, A_q$ in the proof of Theorem 4.2, and the desired OA $M_q$ results.

The following example illustrates the application of Theorem 4.2.

EXAMPLE 4.2 (A new family of $\mathrm{OA}(2^5 p_1 p_2 p_3, 2^{18} p_1^1 p_2^1 p_3^1, 3)$ produced with $p_1$, $p_2$ and $p_3$ being odd primes that are greater than or equal to 5 and not all equal). Step 1. For fixed $r = 1$ and $q = 3$, decompose $n_1 = n_{11} + n_{12} + n_{13} = 1 + 0 + 0$, $n_2 = n_{21} + n_{22} + n_{23} = 0 + 1 + 0$, $n_3 = n_{31} + n_{32} + n_{33} = 0 + 0 + 1$, and $n_4 = n_{41} + n_{42} + n_{43} = 6 + 6 + 6$. For $j = 1, 2, 3$, there exists an $\mathrm{OA}(8p_j, 2^6 p_j^1, 3)$ such that by juxtaposition, and using computer search we can find its strength 1 orthogonal partition $\{A_{0j}, \dots, A_{3j}\}$.

Step 2. Identify the initial $\mathrm{OA}(4, 4^3, 1)$.

Step 3. Using all of the orthogonal partitions $\{A_{0j}, \dots, A_{3j}\}$, the desired new family of OAs can be obtained.

In particular, for $p_1 = p_2 = 5$ and $p_3 = 7$, the example yields an $\mathrm{OA}(2^5 5^2 7^1, 2^{18} 5^2 7^1, 3)$. Let $p_1 = 5$, $p_2 = 7$ and $p_3 = 11$. Then there exists an $\mathrm{OA}(2^5 5^1 7^1 11^1, 2^{18} 5^1 7^1 11^1, 3)$.

As constructed in Examples 3.1 and 3.2, neither of the two new families of OAs in Examples 4.1 and 4.2 can be obtained using previous methods. Moreover, the proposed OAs have more flexible structures.

These examples are introduced only for the purpose of illustrating applications of our methods. The newly constructed arrays are simply a small proportion of what can be obtained. This is summarized in Tables 1 and 2. In Part I in the Supplement Material (Pang et al. (2021)), Tables S1, S2 and S3 provide more detailed information for constructing these new OAs of strengths 3, 4 and $\ge 5$, respectively. Tight OAs are of substantial importance in the design of experiments as optimal fractional factorial plans with the least number of runs. Constructing such OAs and OAs with the maximum numbers of factors is always of high interest. Table S4 in the Supplement Material (Pang et al. (2021)) presents further details about the construction of new tight OAs and OAs with the largest possible numbers of factors.

**5. Discussion and concluding remarks.** A variety of designs resulted from OAs have been recently applied to statistics, combinatorics and theoretical studies for information science and computer science. OAs of high strength are sometimes more useful than OAs of strength 2, as their characteristics allow us to study the interactions between two factors and among three or more factors in the factorial designs. Some statisticians are also concerned with how to use the orthogonality of OAs to deal with big data. However, OAs of high strength, especially asymmetric OAs with factors whose numbers of levels are nonprime powers, are still scarce. How to construct OAs of high strength of the sort required for practical use remains an open problem. Zhang and his coauthors wrote a series of papers on constructing OAs of strength 2 based on orthogonal decompositions of projection matrices (Zhang (2007), Zhang, Lu and Pang (1999) and Zhang, Pang and Wang (2001)). The present paper builds in part on those papers and proposes construction methods for high strength $(t \ge 3)$ OAs based on orthogonal partitions of smaller OAs and spaces. Some of the ideas are similar in facilitating the construction of larger OAs from smaller arrays.

TABLE 1
*Selective newly constructed OAs of strengths 3 and 4*‡

| OAs of strength 3 | OAs of strength 4 |
|---|---|
| $\mathrm{OA}(3^3n, 3^9p_1^{n_1}\cdots p_v^{n_v}, 3)$ | $\mathrm{OA}(2^9p^1, 2^48^3p^1, 4)$ |
| $3\mid n$ | $\frac{p}{4}$ is odd |
| $\mathrm{OA}(2^43^1p, 2^{10}3^1p^1, 3)$ | $\mathrm{OA}(3^{10}p^3, 3^5(27p)^3, 4)$ |
| $p \geq 5$ is a prime | $p$ is an integer |
| $\mathrm{OA}(2^4p, 2^{10}p^1, 3)$ | $\mathrm{OA}(p^6, p^{p+5}, 4)$ |
| $p \geq 5$ is a prime | $p \geq 4$ is a prime power |
| $\mathrm{OA}(2^4p_1p_2, 2^{12}p_1^1p_2^1, 3)$ | $\mathrm{OA}(2^45^4p, 2^55^5(2p)^1, 4)$ |
| $p_1, p_2 \geq 5$ are primes and $p_1 \neq p_2$ | $p$ is odd and $5 \nmid p$ |
| $\mathrm{OA}(2^43^1p^2, 2^76^1p^3, 3)$ | $\mathrm{OA}(2^4p^4, 2^6p^p, 4)$ |
| $p = 3, 6, 12, 24$ | $p = 5, 7, 11$ |
| $\mathrm{OA}(2^63^3p^2, 2^{11}4^1(6p)^3, 3)$ | $\mathrm{OA}(2^5p^4, 2^7p^{p+1}, 4)$ |
| $p = 1, 2, 4, 8$ | $p \geq 5$ is a prime power |
| $\mathrm{OA}(2^{n+3}p, 2^{2+2^{n+2}}p^1, 3)$ | $\mathrm{OA}(p^6, p^4(p^2)^2, 4)$ |
| $p \geq 5$ is a prime and $n \geq 1$ | $\frac{p}{2} > 3$ is an odd prime power |
| $\mathrm{OA}(2^{n+3}5^1, 2^{2+2^{n+2}}5^1, 3)$ | $\mathrm{OA}(p^{s_1+2s_2+2}, p^4(p^{s_1})^1(p^{s_2})^2, 4)$ |
| $n \geq 1$ | $\frac{p}{2}$ $(\geq 5)$ or $\frac{p}{5}$ $(\geq 3)$ is a prime |
| $\mathrm{OA}(2ns^2, 2^ns^{s+1}, 3)$ | $\mathrm{OA}(p^5, p^7, 4)$ |
| $s$ is a power of 2 | $\frac{p}{2}$ $(\geq 5)$ or $\frac{p}{5}$ $(\geq 3)$ is a prime |
| $H_n$ exists and $s\mid 2n$ | $\mathrm{OA}(p^{6+q}, p^{pq+1}, 4)$ |
| $\mathrm{OA}(2^5p_1p_2p_3, 2^{18}p_1^1p_2^1p_3^1, 3)$ | $p \geq 4$ is a prime and $3 \leq q \leq p^3+1$ |
| $p_1, p_2, p_3 \geq 5$ are odd primes | $\mathrm{OA}(2^83^1, 2^44^36^1, 4)$ |
| and not all equal | $\mathrm{OA}(7^6, 7^{13}, 4)$ |
| $\mathrm{OA}(2^33^2, 2^{12}3^2, 3)$ | $\mathrm{OA}(8^6, 8^{15}, 4)$ |
| $\mathrm{OA}(2^43^2, 2^{11}3^24^1, 3)$ | $\mathrm{OA}(8^{10}, 8^8(8^3)^3, 4)$ |
| $\mathrm{OA}(2^55^1, 2^64^25^1, 3)$ | $\mathrm{OA}(2^{16}, 2^{13}8^8, 4)$ |
| $\mathrm{OA}(2^57^1, 2^64^27^1, 3)$ | $\mathrm{OA}(2^53^4, 2^73^5, 4)$ |
| $\mathrm{OA}(2^45^1, 2^{10}5^1, 3)$ | $\mathrm{OA}(2^43^5, 2^83^4, 4)$ |
| $\mathrm{OA}(2^47^1, 2^{10}7^1, 3)$ | $\mathrm{OA}(2^55^4, 2^45^520^1, 4)$ |
| $\mathrm{OA}(2^43^15^1, 2^{10}3^15^1, 3)$ | $\mathrm{OA}(2^55^4, 2^74^15^5, 4)$ |
| $\mathrm{OA}(2^45^2, 2^95^110^1, 3)$ | $\mathrm{OA}(2^47^4, 2^67^7, 4)$ |
| $\mathrm{OA}(2^45^17^1, 2^{12}5^17^1, 3)$ | $\mathrm{OA}(2^47^4, 2^57^8, 4)$ |
| $\mathrm{OA}(2^73^1, 2^93^18^2, 3)$ | $\mathrm{OA}(2^47^4, 2^47^714^1, 4)$ |
| $\mathrm{OA}(2^55^1, 2^{18}5^1, 3)$ | $\mathrm{OA}(2^{13}, 2^58^8, 4)$ |
| $\mathrm{OA}(2^53^2, 2^{16}3^16^1, 3)$ | $\mathrm{OA}(2^{13}5^1, 2^48^810^1, 4)$ |
| $\mathrm{OA}(2^65^1, 2^{34}5^1, 3)$ | $\mathrm{OA}(2^{13}5^1, 2^68^8, 4)$ |
| $\mathrm{OA}(2^13^5, 2^13^{14}, 3)$ | $\mathrm{OA}(2^43^8, 2^63^19^9, 4)$ |
| $\mathrm{OA}(2^75^1, 2^{66}5^1, 3)$ | $\mathrm{OA}(2^43^8, 2^89^9, 4)$ |
| $\mathrm{OA}(2^43^3, 2^73^36^1, 3)$ | $\mathrm{OA}(2^53^8, 2^79^{10}, 4)$ |
| $\mathrm{OA}(2^43^3, 2^43^44^1, 3)$ | $\mathrm{OA}(2^{10}, 2^48^3, 4)$ |
| $\mathrm{OA}(2^45^3, 2^64^15^6, 3)$ | $\mathrm{OA}(14^6, 14^4196^2, 4)$ |
| $\mathrm{OA}(2^37^3, 2^67^8, 3)$ | $\mathrm{OA}(18^6, 18^4324^2, 4)$ |
| $\mathrm{OA}(2^95^1, 2^65^18^9, 3)$ | |
| $\mathrm{OA}(2^{10}3^1, 2^76^18^9, 3)$ | |
| $\mathrm{OA}(2^{10}3^1, 2^93^18^9, 3)$ | |
| $\mathrm{OA}(2^{10}3^1, 2^43^14^18^9, 3)$ | |
| $\mathrm{OA}(2^97^1, 2^{28}8^9, 3)$ | |
| $\mathrm{OA}(2^97^1, 2^67^18^9, 3)$ | |
| $\mathrm{OA}(2^{12}3^1, 2^93^116^{17}, 3)$ | |
| $\mathrm{OA}(2^55^27^1, 2^{18}5^27^1, 3)$ | |
| $\mathrm{OA}(2^55^17^111^1, 2^{18}5^17^111^1, 3)$ | |

‡ Displayed on the website http://web.stat.nankai.edu.cn/mqliu/MOA/MixedOA.html.

TABLE 2
*Selective newly constructed OAs of strength $t \geq 5$, tight OAs and OAs with the largest possible numbers of factors*‡

| OAs of strength $t \geq 5$ | Tight OAs and OAs with the largest possible numbers of factors |
|---|---|
| $\mathrm{OA}(p^{2p}, p^{2(p+1)}, 2p-1)$ | $\mathrm{OA}(2p^2, 2^p p^2, 3)$ |
| $p > 4$ is a prime power | $H_p$ exists and $8 \nmid p$ |
| $\mathrm{OA}(2^{2n+5}3^2, 2^3 4^n 12^2, n+3)$ | $\mathrm{OA}(2^{n+3}p^1, 2^{2+2^{n+2}}p^1, 3)$ |
| $n \geq 2$ | $p \geq 5$ is a prime and $n \geq 1$ |
| $\mathrm{OA}(Np^{t-1}, p^m p_2^{n_2} \cdots p_v^{n_v}, t)$ | $\mathrm{OA}(2^{n+3}5^1, 2^{2+2^{n+2}}5^1, 3)$ |
| if $p$ is a power of 2 and $t = 3$ | $n \geq 1$ |
| $m = p + 1$, otherwise $m = p$ | $\mathrm{OA}(2^4 p, 2^{10}p^1, 3)$ |
| $\mathrm{OA}(p^{2s_1+2s_2+2}, p^4(p^{s_1})^2(p^{s_2})^2, 5)$ | $p \geq 5$ is a prime |
| $\frac{p}{2}$ $(\geq 5)$ or $\frac{p}{5}$ $(\geq 1)$ is a prime | $\mathrm{OA}(2^5 3^2, 2^{12}12^2, 3)$ |
| $\mathrm{OA}(p^9, p^8(p^2)^2, 5)$ | $\mathrm{OA}(2^5 5^2, 2^{20}20^2, 3)$ |
| $p = 2$ or $p$ is not a prime power | $\mathrm{OA}(2^5 7^2, 2^{28}28^2, 3)$ |
| $\mathrm{OA}(4^{s_1+1}9^{s_2}p^2, 2^4(2^{s_1})^2(3^{s_2})^2 p^2, 5)$ | $\mathrm{OA}(2^5 11^2, 2^{44}44^2, 3)$ |
| $p$ is even | $\mathrm{OA}(2^5 3^2 5^2, 2^{60}60^2, 3)$ |
| $\mathrm{OA}(p^8, p^{p+7}, 5)$ | $\mathrm{OA}(2^4 5^1, 2^{10}5^1, 3)$ |
| $p \geq 7$ is a prime power | $\mathrm{OA}(2^5 5^1, 2^{18}5^1, 3)$ |
| $\mathrm{OA}(p^6, p^8, 5)$ | $\mathrm{OA}(2^6 5^1, 2^{34}5^1, 3)$ |
| $\frac{p}{2}$ $(\geq 5)$ or $\frac{p}{5}$ $(\geq 3)$ is a prime | $\mathrm{OA}(2^7 5^1, 2^{66}5^1, 3)$ |
| $\mathrm{OA}(p^{2+3m}, p^{5m}, 7)$ | $\mathrm{OA}(2^4 7^1, 2^{10}7^1, 3)$ |
| when $p = 5, 7$, $m = 2$ | |
| when $p = 5, 7, 9$, $m = 3$ | |
| $\mathrm{OA}(p^{p+5}, p^{2(p+2)}, 7)$ | |
| $p \geq 4$ is a power of 2 | |
| $\mathrm{OA}(p^{p+m+1}, p^{2(p+1)}, 2m+1)$ | |
| $p \geq 4$ is a prime power | |
| and $2 \leq m \leq p - 1$ | |
| $\mathrm{OA}(p^{11+2m}, p^{15+4m}, 7)$ | |
| when $p = 5, 7, 9$, $m = 0, 1$ | |
| when $p = 11$, $m = 1$ | |
| $\mathrm{OA}(p^{10+2m}, p^{12+4m}, 8)$ | |
| when $p = 4, 5, 7$, $m = 0$ | |
| when $p = 4, 5, 7, 9$, $m = 1, 2$ | |
| when $p = 11$, $m = 2$ | |
| $\mathrm{OA}(2^9, 2^3 4^4, 5)$ | |
| $\mathrm{OA}(2^9 3^2, 2^3 4^2 12^2, 5)$ | |
| $\mathrm{OA}(3^8, 3^{10}, 6)$ | |
| $\mathrm{OA}(2^8, 2^6 4^2, 5)$ | |
| $\mathrm{OA}(2^{12}, 2^8 8^2, 6)$ | |
| $\mathrm{OA}(2^{10}, 2^6 8^2, 5)$ | |
| $\mathrm{OA}(2^9, 2^8 4^2, 5)$ | |
| $\mathrm{OA}(2^{17}, 2^{48}, 5)$ | |
| $\mathrm{OA}(2^{11}, 2^{16}8^2, 5)$ | |
| $\mathrm{OA}(2^{11}, 2^{16}4^3, 5)$ | |

‡ Displayed on the website http://web.stat.nankai.edu.cn/mqliu/MOA/MixedOA.html.

It is worth mentioning that the initial OA is one of the smaller arrays in each of the construction methods. By initial, we mean existing and starting. An initial OA, along with other ingredients, is necessarily used as the starting point of our construction. When constructing a new OA, we first need to choose a proper initial OA according to the parameters of the new OA. The methods using initial OAs with $r+1$ columns and of strength $r$ are different from those using initial OAs with $q$ $(> r+1)$ columns and of strength $r$. Sometimes, choosing an initial OA with $r+1$ columns and of strength $r$ to construct a new OA is easier and more feasible than choosing an initial OA with $q$ $(> r+1)$ columns, but at other times this is not the case. The construction methods in Section 3 can be used if someone chooses an initial OA with $r+1$ columns and of strength $r$, otherwise, the methods in Section 4 can be used. Theoretically, any OA including any of the new OAs we have constructed can be used as an initial OA for constructing another new OA.

In this paper, several new construction methods of symmetric and asymmetric OAs with high strength are proposed by using the lower strength orthogonal partitions. As a consequence, we provide a solid answer to Research Problem 9.33 in Hedayat, Sloane and Stufken (1999). Our methods have the following advantages.

1. The variety of spaces, OAs, and orthogonal partitions greatly increases the variety of the asymmetric OAs obtained. Therefore, compared with the existing constructions, the proposed methods have three favorable properties: various strengths, larger sizes and flexibility in the choice of factor levels.

2. It is increasingly difficult to construct the following three families of OAs: symmetric OAs with higher strength, high strength asymmetric OAs with all factor levels being prime powers, and high strength asymmetric OAs with factor levels being nonprime powers. The existing asymmetric OAs $\mathrm{OA}(N, p_1^{n_1} p_2^{n_2} \cdots p_v^{n_v}, t)$ are still scarce when $p_i$ is not a prime power. Compared with Suen, Das and Dey (2001) and Suen and Dey (2003), we can construct a number of asymmetric OAs having factors with nonprime power numbers of levels. For example, $\mathrm{OA}(2^{2n+5}3^2, 2^3 4^n 12^2, n+3)$ for $n \geq 2$, $\mathrm{OA}(4^{s_1+1}9^{s_2}p^2, 2^4(2^{s_1})^2(3^{s_2})^2p^2, 5)$ for an even $p$, $\mathrm{OA}(2^4 3^1 p^2, 2^7 6^1 p^3, 3)$ for $p = 3, 6, 12, 24$, $\mathrm{OA}(2^9 p, 2^4 8^3 p^1, 4)$ for odd $p/4$, $\mathrm{OA}(p^{s_1+2s_2+2}, p^4(p^{s_1})^1(p^{s_2})^2, 4)$ for prime $p/2$ $(\geq 5)$ or $p/5$ $(\geq 3)$. The newly proposed methods are simple and easy to implement. The orthogonal partitions of spaces required for our methods could be obtained using row permutations while the orthogonal partitions of OAs can be obtained mainly using Property 7 on page 5 in Hedayat, Sloane and Stufken (1999) and the difference schemes in Hedayat, Stufken and Su (1996).

3. Our theorems imply that the newly obtained OAs have useful orthogonal partitions. In fact, the proposed methods in Theorems 3.2 and 4.2 are iterative. We can use $\mathrm{OA}_1$ to construct $\mathrm{OA}_2$, and $\mathrm{OA}_2$ to construct $\mathrm{OA}_3$, etc. For example, let $\mathrm{OA}_1 = \mathrm{OA}(40, 2^6 5^1, 3)$. From Theorem 3.2, using $\mathrm{OA}(8, 2^4, 3)$ and an initial $\mathrm{OA}(4, 4^2, 1)$, we can construct a new $\mathrm{OA}_2 = \mathrm{OA}(80, 2^{10}5^1, 3)$. Additionally, as a consequence of Theorem 3.2, the array has an orthogonal partition of strength 1. Combining $\mathrm{OA}_2$ with $\mathrm{OA}(16, 2^8, 3)$ and an initial $\mathrm{OA}(8, 8^2, 1)$, we can construct a new $\mathrm{OA}_3 = \mathrm{OA}(160, 2^{18}5^1, 3)$.

4. Applying our theorems and corollary can lead to many new OAs and their infinite classes. The arrays obtained in this way have higher saturation percentages. Some existing classes of tight arrays and the arrays with the maximal number of columns are easily obtained as special cases. Such OAs are provided in Table S5 in the Supplement Material (Pang et al. (2021)). Note that most tight OAs do not exist for the given parameters. For example, there is only one tight OA among all the 53 OAs with run sizes $\leq 168$ and strength 4 in Schoen, Eendebak and Nguyen (2010). There exist only two tight $\mathrm{OA}(N, s^k, 4)$'s with run sizes $N < 7874496$.

5. The newly constructed families of mixed-level OAs can be useful in design of experiments. Symmetric and asymmetric OAs with strength $t$ are often used for computer experiments in the literature. For example, $\mathrm{OA}(N, p^n, t)$ is used in sliced space-filling designs and nested space-filing designs in Sun, Liu and Qian (2014) and $\mathrm{OA}(N, p_1^{n_1} \cdots p_v^{n_v}, t)$ for sliced Latin hypercube designs in Yin, Lin and Liu (2014). Hedayat, Sloane and Stufken (1999)

showed in Theorem 11.3 that the OA$(N, p_1^{n_1} \cdots p_v^{n_v}, t)$ can be used to estimate main-effects and interactions under the model $Y = XU_1\gamma_1 + XU_2\gamma_2 + \epsilon$. For a design $D$, the generalized word length pattern $(A_1(D), A_2(D), \ldots, A_n(D))$ has a close connection with the strength $t$ of an OA$(N, p_1 \cdots p_v, t)$. The generalized minimum aberration criterion is to sequentially minimize $A_j(D)$ for $j = 1, \ldots, n$ in Jiang and Ai (2017), Xu and Wu (2001) and Zhou and Xu (2014). As stated in Schoen, Eendebak and Nguyen (2010), for OAs with strength 3, the estimates of the main effects are not correlated with interactions between any two other factors, and OAs with strength $t > 3$ can be used to interpret active interaction components. OA$(N, p_1^{n_1} \cdots p_v^{n_v}, t)$ has recently been used in order-of-addition experiments (Peng, Mukerjee and Lin (2019) and Voelkel (2019)). On the other hand, existing asymmetric OAs of high strength are scarce, also limiting their applications. With deep study of their construction methods, we expect that a large number of families of such OAs will be obtained. We believe that they will be more and more widely applied to design of experiments.

In the future, we will investigate a general existence condition of the new families of mixed-level OAs that can be constructed using the proposed methods to help readers decide further whether an OA with particular parameters exists. Since the existence of asymmetric OAs is currently still an open problem, it is of great interest to construct tight OAs or OAs with the maximal number of factors. The results in this paper, especially the results on orthogonal partition of OAs, might provide an important foundation for the construction of this class of OAs. Some of the techniques used in this paper are also useful potentially for studying the existence and construction of other designs.

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## SUPPLEMENTARY MATERIAL

**Supplement to "Construction of mixed orthogonal arrays with high strength"** (DOI: 10.1214/21-AOS2063SUPP; .pdf). The online Supplementary Material contains two sections, where Part I contains Tables S1–S5 and Part II provides all proofs of the lemmas and theorems.

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